

ON LEVIN'S GENERALIZATION OF THE PLUS CONSTRUCTION

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The plus construction was introduced by Kervaire [2] and Quillen [4]. In a simple version it states the following:

Plus Construction Theorem. [1] *Every connected CW complex K with the perfect fundamental group $\pi_1(K)$ is contained in a simply connected CW complex K^+ such that inclusion homomorphisms $H_i(K) \rightarrow H_i(K^+)$ are isomorphisms for all i . Moreover, K^+ can be obtained from K by attaching cells of dimensions 2 and 3.*

Recently Michael Levin found a nice elementary generalization of the Plus Construction Theorem (Proposition 4.4 [3]). As far as I know his generalization was not known before. Since many algebraic topologists refused to believe in this remarkable fact, I decided to write a promotion note.

0.1. Theorem (M. Levin). *For every connected CW complex K there is a simply connected CW complex K^+ obtained from K by attaching cells of dimension 2 and 3 such that the inclusion $K \rightarrow K^+$ induces isomorphisms of homology groups in dimension > 1 .*

Proof. We attached 2-cells to K to obtain a simply connected complex K' . Let $p : K' \rightarrow K'/K$ be the collapsing map. Since $K'/K = \vee S^2$ is the wedge of spheres, the homotopy group $\pi_2(K'/K)$ is free as well as its subgroup $p_\#(\pi_2(K')) = im(p_\#)$. We take a section $s : im(p_\#) \rightarrow \pi_2(K')$ and fix a basis $\{[\phi_j]\}$ for $s(im(p_\#)) \cong \oplus_J \mathbb{Z}$ where $\phi_j : S^2 \rightarrow K'$. Then we attach 3-cells to K' along the maps ϕ_j to obtain K^+ . Note that $H_3(K^+, K') = \oplus_J \mathbb{Z}$ and in view of the Hurewicz isomorphism the through homomorphism

$$H_3(K^+, K') \xrightarrow{\partial} H_2(K') \xrightarrow{p_*} H_2(K'/K)$$

is injective. It takes $H_3(K^+, K') = \oplus_J \mathbb{Z}$ isomorphically onto the image $p_*(H_2(K')) = im(p_*) = \oplus_J \mathbb{Z}$.

Note that the above composition $p_*\partial$ is the connecting homomorphism d in the homology exact sequence of the triple (K^+, K', K) ,

$$\cdots \rightarrow H_3(K', K) \rightarrow H_3(K^+, K) \rightarrow H_3(K^+, K') \xrightarrow{d} H_2(K', K) \xrightarrow{i} H_2(K^+, K) \rightarrow \cdots$$

Then the exact sequence of the triple and the obvious equality $H_3(K', K) = 0$ imply that $H_3(K^+, K) = 0$. Since $H_i(K^+, K) = 0$ for $i > 3$ by dimensional reasons, the homology exact sequence of the pair (K^+, K)

$$\cdots \rightarrow H_{i+1}(K^+, K) \rightarrow H_i(K) \xrightarrow{i'} H_i(K^+) \xrightarrow{j'} H_i(K^+, K) \rightarrow \cdots$$

implies that the inclusion homomorphism $H_i(K) \rightarrow H_i(K^+)$ is an isomorphism for $i \geq 3$ and is a monomorphism for $i = 2$. Also the exact sequence of the triple (K^+, K', K) implies that the inclusion homomorphism $i : H_2(K', K) \rightarrow H_2(K^+, K)$ takes $im(p_*) = im(d)$ to 0. Then the commutative diagram

$$\begin{array}{ccc} H_2(K^+) & \xrightarrow{j'} & H_2(K^+, K) \\ i' \uparrow & & \uparrow i \\ H_2(K') & \xrightarrow{p_*} & H_2(K', K) \end{array}$$

and the fact that i' is surjective for 2-dimensional homology imply that $j' = 0$. Therefore, $H_2(K) \rightarrow H_2(K^+)$ is an epimorphism and, hence, an isomorphism. \square

0.2. Remark. If $\pi_1(K)$ is perfect, hence $H_1(K) = 0$ and Levin's plus construction implies the classical one.

0.3. Remark. The original Levin's proof was presented on the chain level. Here we gave the proof in the diagram chasing language which is perhaps more appealing to algebraic topologists.

0.4. Remark. The original plus construction in the full generality kills a given perfect normal subgroup of $\pi_1(K)$ preserving homology groups. We note that a subgroup version of the plus construction does not admit Levin's type generalization to an arbitrary normal subgroup. As it was noticed in [1] the commutator subgroup of the fundamental group of $K = S^1 \vee S^1$ cannot be killed without creating a new 2-dimensional homology.

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